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Water-wave trapping by floating circular cylinders

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Under the assumptions of the linearized theory of small-amplitude water waves, it is proved that plane waves, normally incident upon a semi-immersed cylinder of uniform circular cross-section floating freely on the surface of a fluid of infinite depth, are capable of being totally reflected. Numerically, this is shown to occur at a single non-dimensional frequency. This remarkable result is used to construct examples of motion-trapped modes, involving pairs of freely floating cylinders moving either in phase or out of phase. The former case is equivalent to having a motion-trapped mode for a single such cylinder next to a rigid vertical wall. In the latter out-of-phase case, the pair of cylinders move as if they form the wetted sections of a single rigidly connected catamaran structure.

1. Introduction

When a body which is floating on the free surface of a fluid which extends indefinitely in a horizontal direction is given a small displacement from its equilibrium position and then released, it is generally assumed that it will oscillate about that position with decreasing amplitude before finally coming to rest. This is because the initial potential energy is expended in creating waves on the free surface which are continually radiated away from the body. In fluid of finite depth the later stages of the motion can be described by a damped harmonic oscillation which is determined by the zero of the so-called complex force coefficient nearest the real axis in the complex frequency plane. The situation in infinitely deep fluid is different however, and Ursell (1964) showed that when a two-dimensional half-immersed circular cylinder is displaced and then released from rest, it makes a finite number of damped oscillations before coming to rest monotonically from below its equilibrium position. Mathematically, this is because the force coefficient in an infinitely deep fluid has a branch cut at the origin of the complex frequency plane which contributes a dominating term for large time which is algebraic in inverse time. The method of solution used by Ursell was to take Fourier transforms of the initial-value problem, thereby converting it into a radiation problem in the frequency or transform domain. The resulting displacement of the cylinder was then given by a Fourier integral over frequency. The denominator of the integrand involved the force coefficient whose imaginary part was related to the damping coefficient for the forced heaving motion of the cylinder, whilst the real part of the denominator involved the added mass of the cylinder in heave.

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Recently, the question has arisen as to whether there exists a structure, in either two or three dimensions, which is free to move in a single degree of freedom, for which the denominator in the Fourier integral describing its displacement vanishes for real values of the transform variable (or frequency). If this were possible, it would mean that when such a structure was displaced from equilibrium and released, it would eventually oscillate indefinitely at that frequency due to the pole of the integrand on the real-frequency axis. Such a structure has been termed by McIver & McIver (2006) a motion-trapping structure and the corresponding localized oscillation of the surrounding fluid a motion-trapped mode to distinguish it from the usual type of trapped mode of a fluid in the vicinity of a fixed body. The vanishing of the denominator provides two conditions for the occurrence of a motion-trapped mode at a particular frequency. First, the wave damping should vanish at that frequency and, secondly, the inertia forces, involving the body's inertia (which includes its added inertia) should balance any spring-restoring force such as the hydrostatic force at the same frequency. It is not difficult to construct shapes in both two and three dimensions for which the radiation damping vanishes at a particular frequency and then the second condition can be satisfied by assuming that an artificial restoring force acts on the body. A recent example of this has been provided by Evans & Porter (2007) who showed that a submerged two-dimensional circular cylinder exhibited zeros of the sway radiation coefficient, and then showed that the second condition could be satisfied by tethering the (buoyant) cylinder appropriately. However, it is more interesting, and also more difficult, to ensure that the second condition is satisfied simultaneously for a freely floating body under natural (hydrostatic) restoring forces.

McIver & McIver (2006), who first derived the conditions described above for a motion-trapping structure moving in a single mode of motion, showed that in deep water, for freely floating structures in heave motion under hydrostatic restoring forces only, the second condition for trapping could be replaced by the requirement that the dipole moment for the potential describing the motion should vanish at infinity. They then constructed a potential from suitably spaced wave sources and dipoles, which was both wave free and had a zero dipole moment at large distances. By sketching the streamlines, they were able to construct 'mirror image' pairs of identical freely floating bodies as trapping structures. Later (McIver & McIver 2007), they were able to extend the idea to construct an axisymmetric freely floating torus of a specific shape which acted as a motion-trapping structure. Recently, Porter & Evans (2008) have used a direct method to show that, for particular frequencies, spacing, thickness and draft, a pair of identical rectangular cylinders in two dimensions could sustain a motion-trapped mode whilst free to make vertical heave motions. The method was extended to thick partly immersed axisymmetric cylinders of rectangular cross-section in an axial plane.

In all cases to date, trapping structures in two dimensions moving under hydrostatic restoring forces have involved pairs of identical 'mirror image' floating cylinders constrained to move in a single mode of motion as if forming the wetted sections of a rigid catamaran hull. Thus, the problem could be replaced by a single cylinder adjacent to a vertical wall on which a Neumann condition is satisfied provided the cylinder was constrained to move in heave only. The question remains as to whether motion-trapped modes exist for an unconstrained cylinder next to a wall, which is free to oscillate in any of its three modes of motion. That is the purpose of this paper. We shall show that a half-immersed circular cylinder of radius a, having its centre a distance b from a rigid vertical wall, and which is free to move in both heave and sway, can oscillate indefinitely without radiating waves to infinity, for particular pairs of values of the parameters Ka and a/b, where $K=\omega^2/g$, ω is the radian frequency of the oscillation and g is gravitational acceleration. An immediate consequence of this result is the non-uniqueness of the scattering problem for a wave incident upon the cylinder and the wall at precisely those parameter values. It might also be supposed that if such a cylinder, correctly positioned with respect to the wall, was given a small vertical displacement and released from rest, it would ultimately oscillate in both sway and heave as a motion-trapped mode, in contrast to the cylinder in the absence of the wall treated by Ursell (1964). This was assumed to be the case in an earlier version of this paper and the authors are grateful to a referee in pointing out our mistake. In fact, the time-dependent problem is much more difficult since the wall induces a transient lateral displacement of the cylinder which prohibits a straightforward Fourier transform solution. Thus, any transients associated with an initial-value problem would have the effect of inducing a time-dependent lateral drift on the cylinder, due to the asymmetric nature of the forces acting upon the cylinder owing to the presence of the wall, and the lack of a natural restoring force in horizontal direction. These transients decay with time after the initial forcing (an initial displacement, or external excitation over a finite period of time) has ceased to act. Since the 'mean' position of the centre of the cylinder is time dependent (there will be small amplitude displacements around the mean which can be dealt with via the usual linearization arguments), the boundary condition that applies on the surface of the cylinder is not amenable to Fourier transforms. Such difficulties are not encountered when a cylinder oscillates in heave only, as the oscillations experienced by the cylinder are about a fixed mean position determined *a priori* by hydrostatics.

In § 2, we consider the frequency domain problem and derive the conditions which need to be satisfied for the cylinder in its coupled motion in heave and sway to exhibit trapped modes confined between the cylinder and the wall. The conditions can be expressed explicitly in terms of the added mass and damping coefficients, including the cross-coefficients for a cylinder in either sway or heave next to a rigid wall. By constraining the heave motion the conditions are shown to reduce to those derived by McIver & McIver (2006) for a trapping structure in a single mode of motion.

The computation of these coefficients for a cylinder in heave or sway next to a wall is non-trivial, and before embarking on the computations it is useful to gather evidence to suggest that motion-trapped modes might actually exist for this configuration. To this end, we consider a much simpler set of related problems involving a single cylinder in the absence of a wall in a similar manner to Porter & Evans (2008). Thus, in § 3, we consider the problem of a plane wave from infinity incident upon the floating cylinder which is free to respond in both heave and sway. Again, because of symmetry, there will be no roll motion. We derive exact explicit formulae for the reflection and transmission coefficients in terms of the added mass and damping of the cylinder in its forced motion in both heave and sway and the reflection and transmission coefficients for the fixed cylinder. In particular, a real condition is derived under which the transmission coefficient vanishes and we are able to demonstrate, using a variety of established asymptotic results for circular cylinders lying in the free surface, that this condition must be satisfied for at least one frequency. This enables a simple wide-spacing approximation to be employed to give an estimate for the wave frequency at which a trapped mode exists between the cylinder and a vertical wall. Note that in this case the conditions rely solely on the added mass and damping coefficients for a cylinder in heave and sway in the absence of the wall, and the reflection and transmission coefficients for the fixed cylinder, all of which are readily computed using Ursell's multipole method (Ursell 1949; Martin & Dixon 1983).

Numerical results connected with the scattering of incident waves by a single freely floating cylinder, including the curves of added mass and damping for cylinders in heave and sway, are presented in § 3. The main results of the paper concerning trapped modes between pairs of cylinders and including curves of added mass and damping coefficients for cylinders next to walls are presented in § 4. The lengthy analytical details associated with the computation of these various hydrodynamic coefficients used to produce the results are contained in a separate technical report available online (see Porter 2008). The work is summarized in § 5.

2. Conditions for a motion-trapped mode

Cartesian coordinates (x, y) are chosen with y downwards and $y = 0$ in the free surface. The fluid of density ρ occupies $y > 0$ outside the cylinder, which has radius a, and centre $(0, 0)$ in equilibrium, and the vertical wall which occupies $x = -b$ on which the normal fluid velocity vanishes.

Classical linear water wave theory is used so that the small-amplitude fluid motion is expressed in terms of a velocity potential being the real part of $\{\Phi^w(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ assuming the motion of the fluid and the structure is time harmonic with angular frequency ω. The time dependence is suppressed for the remainder of the paper. In order to construct a motion-trapped mode, we seek a potential $\Phi^w(x, y)$ describing persistent simple harmonic motion of the cylinder, free to move in heave and sway, at a particular frequency and distance from a vertical wall, on which the normal velocity vanishes, and for which the wave field at large distances vanishes. The components of velocity of the cylinder in the x and y directions are denoted by U_1^w and U_2^w, respectively. The superscript w is used throughout to indicate the presence of the wall.

Then $\Phi^w(x, y)$ is harmonic in the fluid and satisfies

$$\left.\begin{aligned} K\Phi^w + \Phi^w_y &= 0, && \text{on } y = 0,\\ |\nabla\Phi^w| &\to 0, && \text{as } y \to \infty,\\ \Phi^w_x &= 0, && \text{on } x = -b,\\ \Phi^w_r &= U_1^w \sin\theta + U_2^w \cos\theta, && \text{on } r = a, \end{aligned}\right\} \tag{2.1}$$

where $K = \omega^2/g$ (g is gravitational acceleration) and $x = r\sin\theta$, $y = r\cos\theta$ so that $\theta = 0$ coincides with the vertical coordinate axis.

The relationship between velocity and displacement of the cylinder in each component is given by

$$U_j^w(\omega) = -\mathrm{i}\omega X_j^w, \qquad (j = 1, 2), \tag{2.2}$$

whilst the equation of motion in the frequency domain is given by

$$-M\omega^2 X_j^w = -\lambda\delta_{j2}X_j^w + F^w_{R_j}, \qquad (j = 1, 2). \tag{2.3}$$

The first term on the right-hand side represents the hydrostatic restoring force in vertical (heave) motion with

$$\lambda = 2a\rho g, \tag{2.4}$$

whilst $F^w_{R_j}(\omega)$ represents the (complex) hydrodynamic force on the cylinder.

It is clear that, with a suitable radiation condition applied, we can write

$$\Phi^w = U_1^w \Phi_{R_1}^w + U_2^w \Phi_{R_2}^w, \tag{2.5}$$

where $\Phi_{R_1}^w$ (or $\Phi_{R_2}^w$) is the radiation potential due to the forced unit sway (or heave) velocity of the cylinder at the given frequency $\omega/2\pi$. In the far-field, it is assumed

$$\Phi_{R_j}^w \sim A_j^w \mathrm{e}^{\mathrm{i}Kx-Ky}, \qquad x \to \infty. \tag{2.6}$$

Because of the presence of the wall, there are both horizontal and vertical components of the wave force on the cylinder due to its sway (or heave) velocity U_1^w (or U_2^w) and so it follows that in terms of added mass and damping coefficients (see, for example, Newman 1977),

$$F_{R_j}^w = -\sum_{k=1}^{2} \big(b_{jk}^w(\omega) - \mathrm{i}\omega a_{jk}^w(\omega)\big) U_k^w \tag{2.7}$$

in which $a_{12}^w = a_{21}^w$, $b_{12}^w = b_{21}^w$. The following relationships are readily established by use of the functions $\Phi_{R_j}^w$ and $\Psi_k^w \equiv \Phi_{R_k}^w - \overline{\Phi}_{R_k}^w$ (where the overbar denotes complex conjugate) in Green's identity applied to the fluid domain, $x > -b$,

$$b_{jk}^w = \tfrac{1}{2}\rho\omega A_j^w \overline{A}_k^w, \qquad j, k = 1, 2, \tag{2.8}$$

and $A_1^w \overline{A}_2^w = A_2^w \overline{A}_1^w$ is real. This latter identity can be established by application of Green's identity to the potentials Ψ_1^w and Ψ_2^w. Thus, (2.8) shows that b_{jk}^w are real, and b_{jj}^w are non-negative. The pre-factor of 1/2 in (2.8) is not present in (3.10) because waves are radiated to plus infinity only. It is also worth noting the identity

$$b_{11}^w b_{22}^w = b_{12}^w b_{21}^w, \tag{2.9}$$

which follows from (2.8).

Substitution of (2.7) into (2.3) gives

$$\sum_{k=1}^{2} \big(C_{jk}^w + \mathrm{i}\big) b_{jk}^w U_k^w = 0, \tag{2.10}$$

where

$$C_{jk}^w = \big\{\big(M\delta_{jk} + a_{jk}^w\big)\omega^2 - \delta_{j2}\delta_{k2}\lambda\big\}/b_{jk}^w\omega. \tag{2.11}$$

Non-trivial solutions exist when

$$\Delta \equiv \big(C_{11}^w + \mathrm{i}\big)\big(C_{22}^w + \mathrm{i}\big) b_{11}^w b_{22}^w - \big(C_{12}^w + \mathrm{i}\big)^2 b_{12}^w b_{21}^w = 0. \tag{2.12}$$

We can constrain the cylinder to move in heave only by repeating the above analysis and including a restoring force in sway which is then allowed to become infinitely large. This turns out to be equivalent to letting $C_{11}^w \to \infty$. In this case, it follows that $U_1^w \to 0$ and the condition $\Delta = 0$ above becomes

$$\big(C_{22}^w + \mathrm{i}\big) b_{22}^w \omega \equiv \big(M + a_{22}^w\big)\omega^2 - \lambda + \mathrm{i}\omega b_{22}^w = 0 \tag{2.13}$$

and the real and imaginary parts of this equation are the conditions first given by McIver & McIver (2006) for a motion-trapped mode for a structure moving freely in a single (heave) mode of motion.

The condition $\Delta = 0$ in (2.12) is simplified on using (2.9) and splits easily into real and imaginary parts to give $C_{11}^w C_{22}^w = (C_{12}^w)^2$ and $C_{11}^w + C_{22}^w = 2C_{12}^w$ and these combine to give

$$C_{11}^w = C_{22}^w = C_{12}^w \tag{2.14}$$

as the two real conditions to be satisfied simultaneously for a motion-trapped mode.

If we consider (2.10), we obtain

$$\frac{U_1^w}{U_2^w} = -\frac{\left(C_{12}^w + \mathrm{i}\right)b_{12}^w}{\left(C_{11}^w + \mathrm{i}\right)b_{11}^w} = -\frac{b_{12}^w}{b_{11}^w}, \tag{2.15}$$

when (2.14) is used. Thus, when conditions for a motion-trapped mode are met, U_1^w/U_2^w is real, so that U_1^w and U_2^w are in phase and the centre of the cylinder next to the wall moves in time-harmonic motion along a straight line. Furthermore, using (2.8) in (2.15) shows that $U_1^w/U_2^w = -A_2^w/A_1^w$, and so $U_1^w A_1^w + U_2^w A_2^w = 0$. This simply confirms that the far-field amplitude due to the combined heave and sway motions under trapped mode conditions is zero.

These conditions which involve calculations of the added mass and radiation damping terms for cylinders next to walls are non-trivial to formulate and calculate and, at this stage, there is no *a priori* guarantee that they have a real solution.

Thus, before embarking on this route we consider a different problem having an easier solution which, as described in the Introduction, will provide us with an approximate condition for a motion-trapped mode when the wall is some distance away from the heaving and swaying cylinder. Thus, we consider the scattering of a wave incident from infinity by a half-immersed semi-circular cylinder free to move in both heave and sway, and obtain explicit expressions for the associated reflection and transmission coefficients (which we call $\widehat{R}$, $\widehat{T}$) in terms of added mass and damping coefficients for the cylinder in the absence of the wall. This is a much easier problem to approach, and has already been considered by Martin & Dixon (1983). As will be shown, we find there is a single value of the non-dimensional frequency parameter, $K_0 a$ at which $\widehat{T} = 0$, $|\widehat{R}| = 1$ and so total reflection occurs. Thus, at a sufficient distance downstream of the cylinder for local effects to be negligible, the wave field is a standing wave, being the sum of the incident wave plus the reflected wave of unit modulus. We may now insert a rigid wall at a distance $-b$ from the centre of the cylinder and the requirement of no flow through the wall gives the condition

$$\widehat{R} = \mathrm{e}^{-2\mathrm{i}K_0 b}. \tag{2.16}$$

This can be translated into an approximate formula for the cylinder to wall spacing of $a/b = K_0 a/(n\pi - \frac{1}{2}\arg\{\widehat{R}\})$, where n is some integer, large enough to satisfy the geometric constraint $a/b < 1$.

This wide-spacing argument therefore furnishes us with the approximate values of $K_0 a$ and a/b at which we expect to find a motion-trapping structure for a cylinder next to a wall.

3. The scattering of an incident wave by a freely floating circular cylinder

We assume a plane wave of frequency $\omega/2\pi$ is incident from $-\infty$ on the cylinder which responds with the same frequency. Then we may write

$$\Phi = \sum_{j=1}^{2} U_j \Phi_{R_j} + \Phi_S, \tag{3.1}$$

where Φ_S is the scattered potential due to a unit amplitude incident wave on the cylinder assumed to be held fixed, Φ_{R_j} and U_j are radiation potentials and component velocities introduced in the previous section, but now in the absence of the wall. We have

$$\Phi_{R_j} \sim \{\mathrm{sgn}(x)\}^j A_j\, \mathrm{e}^{\mathrm{i}K|x|-Ky}, \qquad \text{as } |x| \to \infty,\ (j=1,2), \tag{3.2}$$

where A_j are far-field radiated wave amplitudes as $x \to \infty$ in sway ($j=1$) and heave ($j=2$), with

$$\Phi_S \sim \begin{cases} (gA/\omega)(\mathrm{e}^{\mathrm{i}Kx-Ky} + R\mathrm{e}^{-\mathrm{i}Kx-Ky}), & x \to -\infty, \\ (gA/\omega)T\mathrm{e}^{\mathrm{i}Kx-Ky}, & x \to \infty, \end{cases} \tag{3.3}$$

where R and T are the reflection and transmission coefficients for the fixed cylinder, dependent on frequency, and A is the prescribed (complex) incident wave amplitude. It follows that

$$\Phi \sim \begin{cases} (gA/\omega)(\mathrm{e}^{\mathrm{i}Kx-Ky} + \widehat{R}\mathrm{e}^{-\mathrm{i}Kx-Ky}), & x \to -\infty, \\ (gA/\omega)\widehat{T}\mathrm{e}^{\mathrm{i}Kx-Ky}, & x \to \infty, \end{cases} \tag{3.4}$$

where

$$\widehat{R} = R + (\omega/gA)\sum_{j=1}^{2}(-1)^j U_j A_j, \qquad \widehat{T} = T + (\omega/gA)\sum_{j=1}^{2} U_j A_j \tag{3.5}$$

are the reflection and transmission coefficients for the freely floating cylinder. Thus,

$$\left.\begin{aligned} \widehat{R} + \widehat{T} &= R + T + 2(\omega/gA)U_2A_2 \\ \widehat{R} - \widehat{T} &= R - T - 2(\omega/gA)U_1A_1 \end{aligned}\right\}. \tag{3.6}$$

The equations of motion in each component are

$$-\mathrm{i}\omega M U_j = F_{R_j} + F_{S_j} + F_j^{ext} \equiv -(b_{jj} - \mathrm{i}\omega a_{jj})U_j + F_{S_j} - \mathrm{i}\delta_{j2}\lambda U_j/\omega, \quad (j = 1, 2), \tag{3.7}$$

where F_{S_j} is the horizontal ($j=1$) and vertical ($j=2$) exciting force on the fixed cylinder which rearranges to

$$b_{jj}(1 - \mathrm{i}C_{jj})U_j = F_{S_j}, \qquad (j = 1, 2), \tag{3.8}$$

where the C_{jj} are defined by (2.11) but without the superscript since the wall is absent here. Now there are a number of reciprocal relations which exist for radiation and scattering problems. For example, the Haskind relation connects the horizontal (or vertical) exciting force on the fixed cylinder to the far-field amplitudes at infinity due to forced unit heave (or sway) velocity of the cylinder. Thus

$$F_{S_j} = (-1)^j \rho g A A_j, \qquad (j = 1, 2). \tag{3.9}$$

Next, we have the relation between the radiation damping coefficient and the far-field amplitude

$$b_{jj} = \rho\omega|A_j|^2, \tag{3.10}$$

and finally the Newman relations

$$R + (-1)^j T = -A_j/\bar{A}_j \equiv -\mathrm{e}^{2\mathrm{i}\theta_j}, \qquad (j = 1, 2), \tag{3.11}$$

connecting the scattering coefficients and the phase of the far-field radiated heave and sway amplitudes. If we make use of these, we find that

$$(-1)^j(\omega/gA)U_jA_j = -(R + (-1)^j T)/(1 - \mathrm{i}C_{jj}), \qquad (j = 1, 2). \tag{3.12}$$

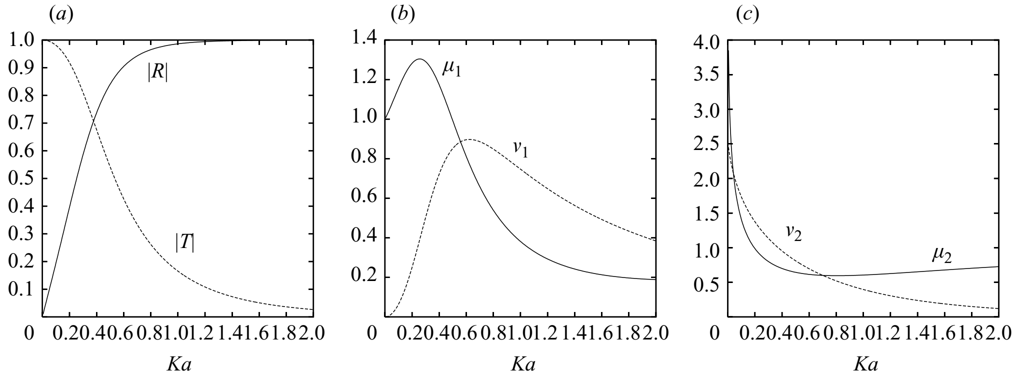

FIGURE 1. Results of the three canonical problems: (*a*) reflection and transmission amplitudes $|R|$, $|T|$ for a fixed cylinder; added-mass and radiation damping coefficients; (*b*) μ_1, ν_1 for a cylinder in forced sway; (*c*) μ_2, ν_2 for a cylinder in forced heave. All as a function of non-dimensional frequency Ka.

Substitution of the above into (3.6) gives

$$\left.\begin{aligned}\widehat{R}+\widehat{T}&=(R+T)(C_{22}-\mathrm{i})/(C_{22}+\mathrm{i})\\ \widehat{R}-\widehat{T}&=(R-T)(C_{11}-\mathrm{i})/(C_{11}+\mathrm{i})\end{aligned}\right\}, \tag{3.13}$$

showing that $|\widehat{R}\pm\widehat{T}|=1$ and $|\widehat{R}|^2+|\widehat{T}|^2=1$, as expected. Thus, we have

$$\left.\begin{aligned}2\widehat{R}&=(R+T)(C_{22}-\mathrm{i})/(C_{22}+\mathrm{i})+(R-T)(C_{11}-\mathrm{i})/(C_{11}+\mathrm{i})\\ 2\widehat{T}&=(R+T)(C_{22}-\mathrm{i})/(C_{22}+\mathrm{i})-(R-T)(C_{11}-\mathrm{i})/(C_{11}+\mathrm{i})\end{aligned}\right\}. \tag{3.14}$$

It follows that $\widehat{T}=0$ provided

$$C_{11}C_{22}+1+(C_{11}-C_{22})\chi=0. \tag{3.15}$$

where $R/T=\mathrm{i}\chi$ and χ is real from the results $|R\pm T|=1$ which arise from symmetry. Since $C_{11}>0$, it is convenient to replace this by

$$f(Ka)\equiv(C_{22}+\chi)+C_{11}^{-1}(1-C_{22}\chi)=0. \tag{3.16}$$

Since χ is real, it follows that $\chi=\pm|R|/|T|$. The sign which χ takes is crucial in what follows, and computations make it clear that for the cylinder, $\chi=-|R|/|T|$ for all values of Ka. This is also true for the scattering by a thin vertical barrier submerged to a depth a first derived by Ursell (1947) where $R/T=\pi I_1(Ka)/\mathrm{i}K_1(Ka)$. Indeed χ, regarded as a function of Ka, can only change sign if there exist values of Ka at which $R=0$ or $T=0$. Again, computations have already shown that this is not the case (see figure 1*a*). Accepting this, it suffices only to show that χ is negative in the limit $Ka\to0$, for example. This can be confirmed analytically by looking at the system of equations which are used to calculate R and T (see Porter 2008). In the limit of small Ka, a leading-order analysis shows that $R\sim-2\mathrm{i}Ka$ and $T\sim1-2\mathrm{i}Ka$ to order Ka and hence that $\chi\sim-2Ka$ to leading order.

It is also insightful to write $C_{jj}=\tan\delta_j$, $(j=1,2)$, with $-\frac{1}{2}\pi<\delta_j<\frac{1}{2}\pi$ and so obtain from (3.15) an alternative form of the condition (3.16) given by

$$\delta_1+\theta_1=\delta_2+\theta_2+n\pi,\quad n\in\mathbb{Z}, \tag{3.17}$$

where (3.11) has been used.

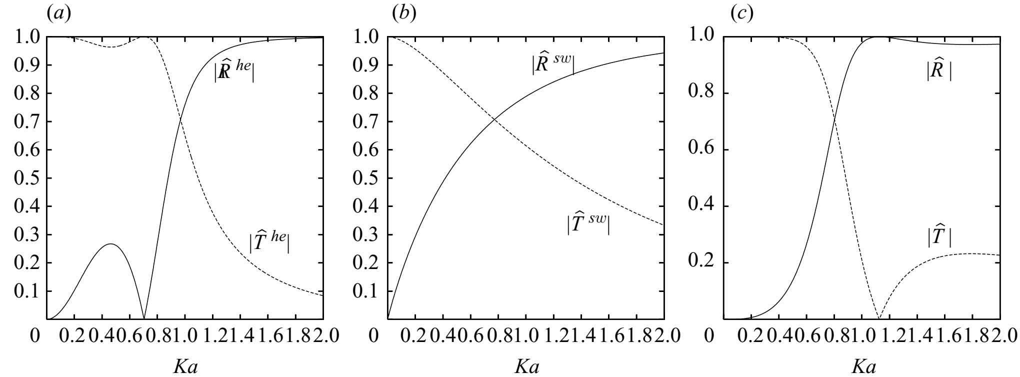

FIGURE 2. Reflected and transmitted wave amplitudes for a cylinder (a) constrained in heave only; (b) constrained in sway only; (c) freely floating. All as a function of non-dimensional frequency Ka.

It is possible to prove that (3.16) does, indeed, have a real solution and hence that there exists a frequency at which $\widehat{T}=0$. To do this, we first non-dimensionalize the added mass and damping coefficients in the usual fashion, by writing

$$a_{jj} = M\mu_j, \qquad b_{jj} = M\nu_j\omega, \quad (j = 1, 2), \tag{3.18}$$

where $M=\frac{1}{2}\pi\rho a^2$ so that, with $\lambda = 2\rho a g$,

$$C_{jj} = (1 + \mu_j - 4\delta_{j2}(\pi Ka)^{-1})/\nu_j, \quad (j = 1, 2). \tag{3.19}$$

A variety of asymptotic results are used in the proof of the result.

First, Ursell (1976, p. 22) states that as $Ka \to 0$, $\mu_1 \sim 1$ and $\nu_1 \to 0$, whilst $\mu_2 \sim -(8/\pi^2)\log Ka$ and $\nu_2 \sim 8/\pi$. These results are highlighted in the curves in figure 1(b–c). Also, as $Ka \to 0$, we have already provided the estimate $\chi \to -2Ka$. It follows that as $Ka \to 0$, $C_{22} \sim -1/(2Ka)$, $C_{11}^{-1} \to 0^+$ and hence $f(Ka) \sim -1/(2Ka)$, $Ka \to 0$.

Next, we go to the opposite limit of $Ka \to \infty$. From Greenhow (1986) we have the results $\mu_1 \sim 4\pi^{-2}$ and $\nu_1 \sim 8/(\pi(Ka)^2)$ whilst $\mu_2 \sim 1 - 4/(3\pi Ka)$ and $\nu_2 \sim 32/(\pi(Ka)^4)$. Also $|R| \to 1$, and Ursell (1961) shows that $|T| \sim 2/(\pi(Ka)^4)$ as $Ka \to \infty$. It follows that $C_{22} \sim \frac{1}{16}\pi(Ka)^4$ and $C_{11}^{-1} \sim 8\pi/((4+\pi^2)(Ka)^2)$ and hence that $f(Ka) \sim \frac{1}{4}\pi^3(Ka)^6(\pi^2+4)$ as $Ka \to \infty$.

Thus, $f(Ka)$ changes sign and there must exist a value of $K = K_0$, say, for which $f(K_0 a) = 0$. This remarkable result is confirmed by numerical calculations which show that there is just one value of $Ka = K_0 a \equiv \omega_0^2 a/g \approx 1.12593$ satisfying (3.15) or, alternatively, (3.17) with $n = 0$, when the principal arguments are used to define θ_j and δ_j. The zero of transmission can be seen in figure 2(c).

We can consider the motion of a cylinder constrained to move in heave only, by letting $C_{11} \to \infty$ (as previously discussed). Then (3.15) ensuring total reflection is modified to $f(Ka) = C_{22} + \chi = 0$. By using the asymptotic expressions previously derived for C_{22} and χ, it is seen that $f(Ka) < 0$ as $Ka \to 0$ and that $f(Ka) < 0$ also in the limit as $Ka \to \infty$. Thus, there is no change of sign in f and therefore no guarantee of a solution of this equation. Taking the limit $C_{11} \to \infty$ in (3.14) gives

$$\widehat{R}^{he} = \mathrm{i}T(C_{22}\chi - 1)/(C_{22} + \mathrm{i}), \qquad \widehat{T}^{he} = T(C_{22} + \chi)/(C_{22} + \mathrm{i}) \tag{3.20}$$

as the reflection and transmission coefficients in heave only. These expressions are identical to those appearing in Porter & Evans (2008) and, previously, Evans & Linton (1989). Numerically generated curves of $|\widehat{R}^{he}|$ and $|\widehat{T}^{he}|$ against Ka are given in figure 2(a), where it is confirmed that there are no zeros of transmission. In contrast, figure 2(a) exhibits a zero of reflection, a common feature of many water-wave problems.

In a similar fashion as before, we can also consider a cylinder constrained to move in sway only by letting $C_{22} \to \infty$. Then the limiting form of (3.14) is

$$\widehat{R}^{sw} = \mathrm{i}T(C_{11}\chi + 1)/(C_{11} + \mathrm{i}), \qquad \widehat{T}^{sw} = T(C_{11} - \chi)/(C_{11} + \mathrm{i}) \tag{3.21}$$

as the reflection and transmission coefficients in sway only. Also see Evans & Linton (1989). From either the above equation or (3.14), the condition for total reflection in sway reduces to $f(Ka) = C_{11} - \chi = 0$. In both limits $Ka \to 0$ and $Ka \to \infty$, the function f tends to plus infinity, and again there is no guarantee of a solution of this equation. The absence of a zero of transmission is confirmed numerically in figure 2(b).

The velocities of the cylinder are given by (3.12) which may be written as

$$(-1)^j U_j |A_j| = (Ag/\omega)\mathrm{e}^{\mathrm{i}(\theta_j + \delta_j)} \cos\delta_j, \qquad (j = 1, 2). \tag{3.22}$$

It follows that

$$U_1/U_2 = -\frac{|A_2| \cos\delta_1}{|A_1| \cos\delta_2} \mathrm{e}^{\mathrm{i}(\theta_1 + \delta_1 - \theta_2 - \delta_2)}. \tag{3.23}$$

Now at the frequency at which total reflection occurs,

$$U_1/U_2 = -\frac{|A_2| \cos\delta_1}{|A_1| \cos\delta_2}, \tag{3.24}$$

where (3.17) with $n = 0$ has been used. The realness of (3.24) implies that when total reflection occurs, the cylinder oscillates along a straight line. Numerically we find $U_1/U_2 = -0.5674$, so that cylinder moves along a line inclined at an angle 29.57° to the vertical.

4. Results

We start by giving some more details to the results mentioned briefly in § 3, relating to the scattering of incident waves by a freely floating semi-immersed circular cylinder. These results rely upon the quantities R, T, μ_j and ν_j ($j = 1, 2$), properties of the solution to three canonical problems namely the scattering of waves by a fixed cylinder and the radiation of waves by the forced motions (of unit velocity) in sway and in heave. The solution method for these three problems can be found in the appendix of Martin & Dixon (1983) and is repeated in the technical report of Porter (2008). Thus, the potential corresponding to waves generated by the cylinder is expanded in a combination of a source, a horizontal dipole and an infinite series of wave-free potentials, following Ursell (1949), and all expressed in local polar coordinates. The application of the cylinder boundary condition yields an infinite system of algebraic equations which are truncated to produce numerical results.

Thus, figure 1(a–c) shows the variation of these key quantities with Ka, the single non-dimensional parameter in this problem. The reflection and transmission coefficients for a freely floating cylinder are now given in terms of these quantities by (3.14) and for cylinders in constrained heave and sway motions by (3.20) and

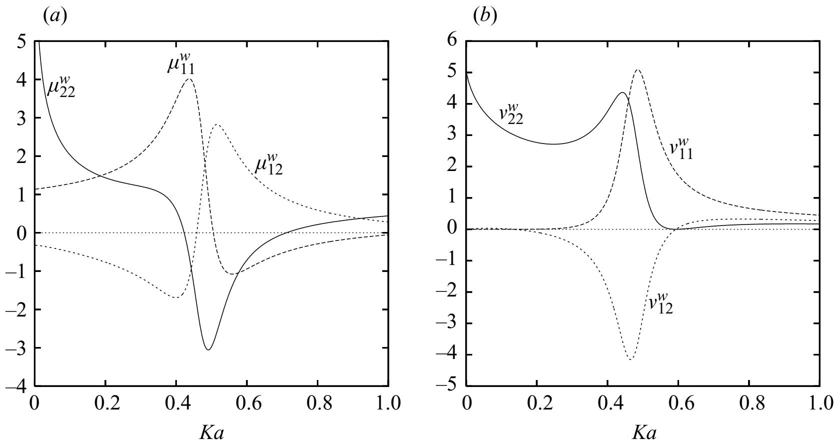

FIGURE 3. Variation of non-dimensional (*a*) added mass and (*b*) radiation damping coefficients for a cylinder next to a wall as a function of Ka for $a/b = 1/2$.

(3.21) and are presented in figure 2(*a–c*). The key result of § 3 is that a semi-immersed cylinder allowed to respond freely in combined heave and sway will reflect all incident wave energy at a particular angular frequency ω_0 where $K_0 a \equiv \omega_0^2 a/g \approx 1.12593$. At this frequency, the cylinder oscillates along a straight line inclined at approximately 29.57° to the vertical.

This result allows us to return to § 2, in which we considered the trapping of waves between a cylinder and a wall, a distance b from the cylinder. The wide-spacing arguments at the end of § 2 give approximate formula for values for $K_0 a$ and a/b at which this is expected to occur. In order to determine exact parameter values at which motion-trapped modes occur, we now need to be able to calculate non-dimensional added mass and radiation damping coefficients $\mu_{jk}^{w} = a_{jk}^{w}/M$ and $\nu_{jk}^{w} = b_{jk}^{w}/M\omega$ (for a cylinder in the presence of a wall) induced in the heave ($k = 1$) and sway ($k = 2$) directions due to the forced motion of unit velocity in heave ($j = 1$) and sway ($j = 2$) directions.

The method of solution here is much more complicated than for a single cylinder. The presence of the wall along $x = -b$ can be accounted for by placing an image cylinder at $x = -2b$ so that the resulting pair moves symmetrically about the line $x = -b$. Now the radiation potentials due to forced sway and heave motions of unit velocity are expanded in terms of sources, horizontal dipoles and wave-free potentials about each of the pair of cylinders. But the application of the boundary condition on the cylinder at the origin requires the singularities from the image cylinder at $(x, y) = (-2b, 0)$ to be expanded in terms of polar coordinates centred at the origin. Whilst this turns out to be a relatively trivial exercise for the wave-free potentials, some care is needed when translating the coordinates in the source and dipole singularities. The details of this procedure are rather lengthy and have been relegated to the online technical report of Porter (2008).

A typical set of results showing the variation of the hydrodynamic coefficients for a cylinder next to a wall for a particular value of $a/b = 1/2$ are presented in figure 3(*a–b*). Note that ν_{jj}^{w} are by definition non-negative and that whenever ν_{12}^{w} crosses the zero axis, one of ν_{11}^{w} and ν_{22}^{w} is simultaneously zero on account of the relation (2.9). It is well known that added mass coefficients can take negative values (see Falnes & McIver 1985).

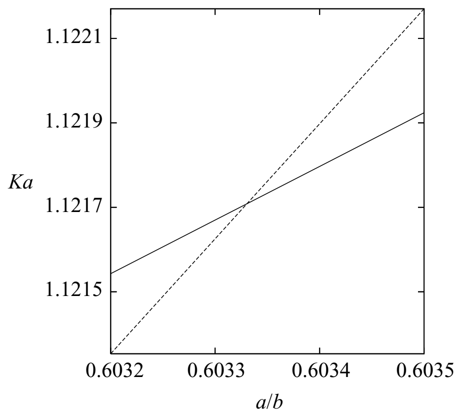

FIGURE 4. Curves of $C^w_{11} - C^w_{12} = 0$ (the solid curve) and $C^w_{22} - C^w_{12} = 0$ (the dashed curve) in $(Ka, a/b)$-parameter space. The crossing point gives exact parameter values for a motion-trapped mode.

We use these calculations of the hydrodynamic coefficients to numerically determine the existence of motion-trapped modes, which correspond to simultaneously satisfying the pair of real conditions given in (2.14). The wide-spacing approximation (2.16) predicts motion-trapped modes for a cylinder next to a wall with $K_0a = 1.12593$ (this is the fixed frequency parameter at which total reflection occurs for a single cylinder) and the sequence $a/b = 0.60484, 0.22505, \ldots, (n = 1, 2, \ldots)$ and these values are used as initial guesses to the exact trapping parameters.

The exact parameters are detected by plotting curves in the $(Ka, a/b)$-plane along which the two real quantities $C^w_{11} - C^w_{12}$ and $C^w_{22} - C^w_{12}$ vanish. A motion-trapped mode corresponds to the crossing of these two curves (as illustrated in figure 4). Although the computation of the curves in figure 4 is approximate, since it involves the numerical truncation of an infinite system of equations, the results are nevertheless computed to at least six significant figure accuracy. Moreover, the intersection of the two curves is robust to changes in numerical accuracy and hence the results provide compelling numerical evidence for the existence of motion-trapped modes.

The parameter values found for exact motion-trapped modes are summarized in table 1. So far we have focused our discussion on cylinders next to walls on which a Neumann condition is enforced. These are equivalent to a mirror-image pair of cylinders moving symmetrically about the line $x = -b$. These are termed symmetric modes in table 1 and there is an infinite sequence of modes. As the mode number increases the distance between the cylinder and the wall increases (or a/b decreases) and the wide-spacing approximation, unsurprisingly, becomes ever more accurate. Additionally, in table 1 we show the real-valued ratio of sway to heave cylinder velocities for a cylinder next to a wall, as given by (2.15), alongside the wide-spacing approximation as given by (3.24) when incident waves are being totally reflected by a cylinder in isolation.

Snapshots in time of the displacements of the wetted sections of the pair cylinders and (on the same scale) the free surface for the first symmetric mode are shown in figure 5. The amplitude of motion is arbitrary, though the linearized theory applies to infinitesimal motions, which accounts for the disjointness between the cylinder boundary and the free surface in the sketch in figure 5.

Instead, we could have placed a Dirichlet boundary condition on the potential on the line $x = -b$ (i.e. $\phi^w(-b, y, t) = 0$). In turn, this would correspond to a pair

	Exact (wide spacing)		
Mode type	K_0a	a/b	U_1/U_2
1st symmetric	1.12170 (1.12593)	0.60333 (0.60484)	−0.57206 (−0.56742)
1st antisymmetric	1.12612 (1.12593)	0.32808 (0.32804)	−0.56702 (−0.56742)
2nd symmmetric	1.12590 (1.12593)	0.22504 (0.22505)	−0.56746 (−0.56742)

TABLE 1. Table showing exact parameters and the corresponding approximations made under the wide-spacing arguments in parentheses against mode type.

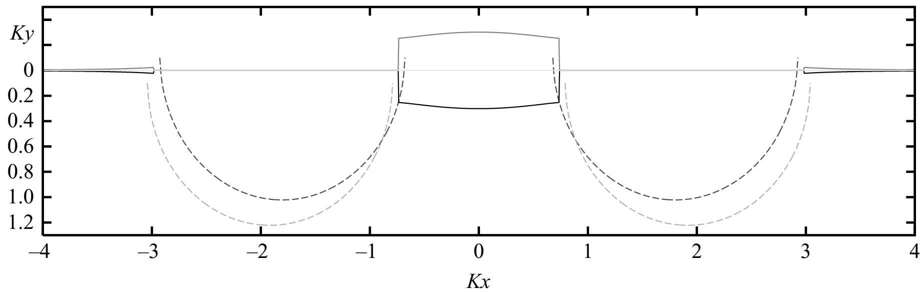

FIGURE 5. Animation of displacements of the free surface (solid lines) and the pair of cylinders (dashed) for the fundamental symmetric mode. Grey curves are π/ω_0-radians advanced in time of black curves, and represent the extremes of the motion.

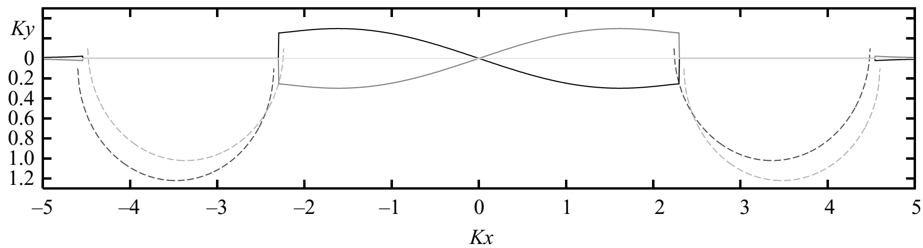

FIGURE 6. Animation of displacements of the free surface (solid lines) and the pair of cylinders (dashed) for the first antisymmetric mode. Grey curves are π/ω_0-radians advanced in time of black curves, and represent the extremes of the motion.

of cylinders moving in anti-phase with respect to each other about the line $x=-b$. Whilst a Dirichlet boundary condition on the fluid has no physical interpretation, two cylinders oscillating in anti-phase move as if connected rigidly above the waterline to form a single catamaran-type structure. There is no mathematical difficulty in replacing the Neumann boundary condition on $x=-b$ to a Dirichlet condition throughout the preceding analysis. Indeed, the change of boundary condition only alters the calculation of the hydrodynamic coefficients in a trivial way (Porter 2008) in addition to modifying the wide-spacing formula to

$$\frac{a}{b}=K_0a\bigg/\left(\left(n+\frac{1}{2}\right)\pi-\frac{1}{2}\arg\{\widehat{R}\}\right).$$

Motion-trapped modes have been determined numerically in this case also and the parameters for the first 'antisymmetric' mode are included in table 1.

The first antisymmetric mode cylinder and free surface displacements are shown in figure 6, where it can be seen that the pair of cylinders form the wetted sections of a single structure which undergoes simultaneous swaying and rolling motion about its mean position.

5. Conclusions

We have shown that pairs of semi-immersed circular cylinders free to move in both heave and sway under natural hydrostatic restoring can support-trapped waves. Underpinning the existence of these examples of motion-trapped waves is the ability for a single semi-immersed circular cylinder freely floating on the surface of the fluid to totally reflect the incoming wave energy at a single frequency. This fact has not only been demonstrated numerically but also proved using a variety of asymptotic results.

These solutions have been shown to exist in the frequency domain and represent a non-uniqueness in a corresponding forced problem. It is not clear how motion-trapped modes may be excited in an initial-value problem for the reasons discussed in the Introduction. However, a continual forced oscillation from time zero at the motion-trapped mode frequency and about the correct mean position from the wall would eventually not radiate waves to infinity.

Given the examples presented both here and in Porter & Evans (2008), it seems highly likely that many more examples of motion-trapped modes can be found between pairs of cylinders of more general cross-section floating in the free surface. This could be done by identifying single cylinders capable of totally reflecting incident waves in free response to the waves and placing two such cylinders at an appropriate distance apart. There is no instance to the authors' knowledge in which this procedure has failed to produce examples of trapped modes between either fixed or floating structures. The example considered here of a cylinder of circular cross-section has the advantage that the roll component is zero, so that only two independent modes of motion need be considered. For other shapes of cylinder in free response to the waves, the roll component would be required and the theory presented here developed to incorporate that extra degree of motion. One would also, almost inevitably, be required to use more general numerical methods in such an investigation. Here, of course, we have taken advantage of the geometry to use solution methods particular to circular cylinders.

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